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On the Rate of Entropy Production for the Boltzmann Equation

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We show that there exists a wide class of distribution functions (with moments of any order as close to their equilibrium values as we like) which can provide an abnormally low rate of entropy production. The result is valid for the Boltzmann equation with any cross section $\sigma(|\mathbf{V}|, \theta)$ satisfying a mild restriction. The functions are constructed in an explicit form and we discuss some applications of our results.

KEY WORDS: Entropy production; Boltzmann equation; kinetic theory.

1. INTRODUCTION

In the last two decades the theory of the space homogeneous Boltzmann equation has achieved many interesting results.(1–9) On one hand, we practically have a rather complete existence and uniqueness theory; on the other hand the research on the qualitative properties of solutions, such as existence of moments, have been studied in some detail. In this paper we address a conjecture put forward by one of the authors(10) many years ago. This conjecture will be introduced below. If true, it would simplify many proofs of the trend to equilibrium in a significant way. In fact one of the corollaries would be that the H-functional tends to a Maxwellian equilibrium exponentially in time. The assumptions originally considered for the initial data were those of finite mass, energy and entropy. In this form the conjecture was disproved by the other author(2) in the case of Maxwellian molecules and by Wennberg(7) for hard spheres.

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Let us consider the matter in more detail. The spatially homogeneous Boltzmann equation reads as follows[8, 9]

$$\partial_t f(\mathbf{v}, t) = Q(f, f)(\mathbf{v}, t) \tag{1.1}$$

where t is time and $Q(f, f)$ the so-called collision operator,

$$Q(f, f)(\mathbf{v}) = \int_{\mathfrak{R}^3} \int_{S^2} (f' f'_* - f f_*)\, B(\theta, |\mathbf{v} - \mathbf{v}_*|)\, d\boldsymbol{\omega}\, d\mathbf{v}_*$$

Here $f = f(\mathbf{v})$, $f_* = f(\mathbf{v}_*)$, $f' = f(\mathbf{v}')$ and $f'_* = f(\mathbf{v}'_*)$; $\mathbf{v}'$ and $\mathbf{v}'_*$ are velocities before the collision of two particles which have the velocities $\mathbf{v}$ and $\mathbf{v}_*$ after they collided. The relation between these velocities are

$$\mathbf{v}' = \frac{\mathbf{v} + \mathbf{v}_*}{2} + \frac{|\mathbf{v} - \mathbf{v}_*|}{2}\, \boldsymbol{\omega}$$

$$\mathbf{v}'_* = \frac{\mathbf{v} + \mathbf{v}_*}{2} - \frac{|\mathbf{v} - \mathbf{v}_*|}{2}\, \boldsymbol{\omega}$$

$\boldsymbol{\omega}$ is the unit vector in the direction from $(\mathbf{v} + \mathbf{v}_*)/2$ to $\mathbf{v}'$, $\cos\theta = \boldsymbol{\omega} \cdot (\mathbf{v} - \mathbf{v}_*)\, |\mathbf{v} - \mathbf{v}_*|^{-1}$. The collision operator is thus an average over all possible collisions that can take place, and B is a weight giving the effect of a particular collision, related to the cross-section $\sigma(|\mathbf{V}|, \theta)$ by $B = |\mathbf{V}| \sigma$. When the particles are hard spheres, $B = |\mathbf{v} - \mathbf{v}_*|$, except for a constant factor.

The theory of the Boltzmann equation is treated e.g., in refs. 8 and 9. The equation is known to be well posed under very general conditions on the initial data. Each collision conserves mass, momentum and energy, and in terms of the solutions f, this translates into the fact that the first moments

$$\int_{\mathfrak{R}^3} f(\mathbf{v}, t)\, d\mathbf{v}, \qquad \int_{\mathfrak{R}^3} f(\mathbf{v}, t)\, \mathbf{v}\, d\mathbf{v}, \qquad \text{and} \qquad \int_{\mathfrak{R}^3} f(\mathbf{v}, t)\, |\mathbf{v}|^2\, d\mathbf{v} \tag{1.2}$$

i.e., the total mass, momentum and energy of the gas, are constant in time. The stationary solutions are the Maxwellians, i.e., functions of the form $a \exp(-|\mathbf{v} - \mathbf{v}_0|^2/(2T))$, and there is a unique Maxwellian corresponding to the conserved quantities Eq. (1.2); it is known that the solutions f converge strongly to this Maxwellian. Since the moment associated with $\mathbf{v}_0$

is constant in time, we can assume $\mathbf{v}_0 = 0$ without any loss of generality. The trend to a Maxwellian is related to the fact that the entropy,

$$H(f) = \int_{\mathfrak{R}^3} f(\mathbf{v}) \log(f(\mathbf{v}))\, d\mathbf{v} \tag{1.3}$$

is monotonously decreasing.

The relative entropy with respect to M is defined as

$$\Delta(f) = \int f(\mathbf{v}) \log \frac{f(\mathbf{v})}{M(\mathbf{v})}\, d\mathbf{v} \tag{1.4}$$

and the entropy production term is

$$D(f) = -\int Q(f, f)(\mathbf{v}) \log f(\mathbf{v})\, d\mathbf{v} \tag{1.5}$$

The entropy production term is positive, and vanishes if and only if f is a Maxwellian,[8, 9] and the relative entropy vanishes if and only if $f = M$. The conjecture mentioned above[10] is that an inequality of the type

$$D(f) \geqslant \lambda\, \Delta(f) \tag{1.6}$$

should hold, with λ depending only on the collision kernel B, and on the Maxwellian M. An estimate of that type would imply, as mentioned above, that the solutions of the Boltzmann equation converge exponentially to equilibrium at a rate depending only on the mass and energy of the initial data. The inequality would also be useful in the study of various limit problems for the full Boltzmann equation.

A first counterexample to the conjecture by Cercignani was indirectly provided by Bobylev[2] for the case of Maxwellian molecules (B independent of $|\mathbf{V}|$). He constructed initial data for Eq. (1.1) which tend to equilibrium exponentially, but at an arbitrarily slow rate. Wennberg[7] studied the relative entropy and the entropy production directly and gave an example showing that, also in the case of hard potentials (such as hard spheres) the inequality cannot be as general as conjectured. Thus it would be interesting to know whether the inequality proposed by Cercignani[10] holds for a smaller class of functions. This question has been partially answered by Carlen and Carvalho (e.g., ref. 4), who proved that for all f with $\int f(\mathbf{v})\, |\mathbf{v}|^s\, d\mathbf{v} < C < \infty$, $s > 2$ there is a strictly increasing function ϕ such that

$$D(f) \geqslant \phi(\Delta(f)) \tag{1.7}$$

Wennberg,[7] concerning (1.7), comments that, "though this has not been proven, it is conceivable that ϕ grows linearly near the origin." It is to be remarked, however, that the condition of uniform boundedness of all moments is quite strong and is certainly not satisfied by Wennberg's counterexample.

One can easily prove that the conjecture by Cercignani[10] is not true even for distribution functions possessing infinitely many moments. This result can be obtained indirectly by the circumstance that, as shown by Desvillettes[5] and Wennberg,[7] a distribution function f_0 possessing finite mass, energy and entropy at time $t=0$ will possess all the moments at an arbitrarily small positive time. Then if the conjecture were true for the case of a distribution with infinitely many moments, by using the said result for a distribution function possessing finite mass, energy and entropy at $t=0$, we could apply the conjecture for infinitely many moments at any time $t=t_0>0$ and find a contradiction. However the Wennberg-type examples (the distribution function f is pointwise close to one Maxwellian M_1, whereas its second moment is close to the second moment of another Maxwellian M_2) imply usually that moments of order $s>2$ grow unboundedly, as f tends to M_1. Therefore, as remarked by Wennberg himself, the examples are not in contradiction with the assumption "that ϕ grows linearly near the origin" in the inequality (1.7).

In the present paper we construct a family of distribution functions (with moments of any order close to their equilibrium values as much as we like) which provide an abnormally low rate of entropy production. One obvious consequence of our results is that the function ϕ in (1.7) cannot grow linearly near the origin.

2. SOME PRELIMINARY CONSIDERATIONS

We assume that the differential cross section is such that

$$g(|\mathbf{V}|)) = 2\pi \, |\mathbf{V}| \int_{-1}^{+1} d\mu \sigma(|\mathbf{V}|, \mu) \leqslant g_\gamma (1 + |\mathbf{V}|^\gamma) \tag{2.1}$$

where g_γ is a constant. This assumption includes all the interesting cases (except soft and long range potentials). The nonphysical case $1<\gamma\leqslant 2$ is also included for the sake of completeness.

By introducing the notation

$$\langle f, g \rangle = \int_{\mathfrak{R}^3} f(\mathbf{v}) \, g(\mathbf{v}) \, d\mathbf{v} \tag{2.2}$$

we can rewrite Eqs. (1.2)–(1.4) as follows:

$$H(f) = \langle f, \log f \rangle \tag{2.3}$$

$$\Delta(f) = H(f) - H(M) \tag{2.4}$$

$$D(f) = -\langle Q(f, f), \log f \rangle \tag{2.5}$$

To clarify the relation between $D(f)$ and $\Delta(f)$, we recall[8, 9] that

$$\frac{\mathrm{d}}{\mathrm{d}t} \Delta(f) = -D(f) \tag{2.6}$$

for any solution of the space-homogeneous Boltzmann equation (1.1).

The classical inequalities:

$$D(f) \geqslant 0, \qquad \Delta(f) \geqslant 0 \tag{2.7}$$

play a fundamental role in kinetic theory.[8, 9] Roughly speaking, they tell us that the solutions of Eq. (1.1) tend to $M(\mathbf{v})$ (as $t \to \infty$) and that the convergence of $\Delta(f)$ to $\Delta(M) = 0$ is monotonous in time.

The above inequalities, however, contain no information about the rate of convergence. If we assume that a stronger inequality, of the form (1.6), holds, then we easily obtain an exponential estimate for the convergence

$$0 \leqslant \Delta(f) \leqslant \Delta(f_0)\, \mathrm{e}^{-\lambda t} \tag{2.8}$$

Moreover, the well-known implication[8, 9]

$$\Delta(f) \to 0 \Rightarrow \|f - M\| \to 0 \tag{2.9}$$

can be used to estimate the convergence $f(\mathbf{v}, t)$. As mentioned in the introduction, the inequality appearing in (1.6), first conjectured by Cercignani,[10] cannot be true, even if we assume that λ depends on infinitely many moments of the distribution function.

In any case, the problem of estimating the rate of entropy production is very important and technically difficult. As pointed out in ref. 10, we must study the functional

$$\Lambda(f) = \frac{D(f)}{\Delta(f)} \tag{2.10}$$

in certain classes of functions and find lower and upper bounds of $\Lambda(f)$. A first difficulty is provided by the complexity of the Boltzmann collision operator $Q(f, f)$, the next one by the fact that $D(M) = \Lambda(M) = 0$. Accordingly, it is very important to understand the behavior of $\Lambda(f) = 0$ in a small neighborhood of M.

Before introducing the main tools used in this paper, we remark that the conjecture (1.6) was partly motivated by the fact that the functional $\Lambda(f)$ is bounded from below on the set

$$\Omega_\varepsilon = \{ f(\mathbf{v}) \geqslant 0, \; \|(f - \mathrm{e}^{-|\mathbf{v}|^2})\, \mathrm{e}^{|\mathbf{v}|^2/2}\|_{L_2} < \varepsilon \} \tag{2.11}$$

provided we can assume that $f - \mathrm{e}^{|\mathbf{v}|^2}$ is of order ε and discard terms of higher order. Thus the conjecture seems still to be probable, even for strongly nonlinear deviations from M provided we consider functions f bounded in the above (weighted) L^2 norm. On the contrary small deviations from M in (weighted) L^1 norms may lead to a violation of the conjectured inequality.

Our knowledge of controls on the entropy source is, however, very scanty if we consider perturbations of M small in some norm, but unbounded in the above L^2 norm. Let us denote by ψ a perturbation of M and assume that it is small, e.g., with respect to the following norm:

$$\|\psi\|_p = \langle \mathrm{e}^{|\mathbf{v}|^{2p}}, \psi \rangle, \qquad 0 < p < 1 \tag{2.12}$$

Our goal is exactly to establish some properties of $\Lambda(f)$ in a small neighborhood of M (with due account for the nonlinearity of $Q(f, f)$), for a wide class of perturbations ψ, which are unbounded in the above L^2 norm but small with respect to the norm (2.12).

3. FORMULATION AND APPLICATION OF THE MAIN RESULTS

We introduce a class of functions $\phi(\mathbf{v})$, satisfying the following simple conditions:

$$\phi(\mathbf{v}) \geqslant a \exp(-b\, |\mathbf{v}|^2), \qquad \|\phi(\mathbf{v})\|_{L^\infty} = \operatorname*{ess\,sup}_{\mathbf{v} \in \mathfrak{R}^3} \phi(\mathbf{v}) < \infty \tag{3.1}$$

for some positive a and b;

$$\langle \phi(\mathbf{v}), 1 + |\mathbf{v}|^{2+\gamma} \rangle < \infty \tag{3.2}$$

where γ $(0 \leqslant \gamma \leqslant 2)$ is the same as in (2.1).

The idea is to fix any such function $\phi(\mathbf{v})$ and consider a two-parameter family of perturbed Maxwellian distributions:

$$f(\mathbf{v}; \varepsilon, v_0) = \mathrm{e}^{-|\mathbf{v}|^2} + \varepsilon\phi(\mathbf{v}/v_0) \tag{3.3}$$

for different (small) values of ε and (large) values of v_0.

Some results concerning the functional $\Lambda(f(\cdot; \varepsilon, v_0))$ depend very weakly on the particular function $\phi(\mathbf{v})$, as we shall presently see. In particular, for values of the scaling parameters ε and v_0 related by:

$$\varepsilon = \exp(-zv_0^{2-\gamma}), \qquad z = \text{const.} \tag{3.4}$$

the following result holds:

Theorem 1. Let $\phi(\mathbf{v})$ be an arbitrary function satisfying (3.1)–(3.2). Then, for any fixed $z > 0$ and $v_0 \to \infty$, the following asymptotic inequality holds:

$$\Lambda(\mathrm{e}^{-|\mathbf{v}|^2} + \phi(\mathbf{v}/v_0)\exp(-zv_0^{2-\gamma})) \leqslant c_\gamma k(\phi)\, z + O(1/v_0)$$
$$c_\gamma = 4\pi^{3/2} g_\gamma,\ k(\phi) = \langle \phi, |\mathbf{v}|^\gamma\rangle / \langle \phi, |\mathbf{v}|^2\rangle \tag{3.5}$$

provided that $0 < \gamma < 2$. If $\gamma = 0$ ($\gamma = 2$), then the above inequality holds only for small (large, respectively) values of z.

Remark 1. It is clear that the perturbation parameter ε is small for $\gamma = 2$ only for sufficiently large values of z. Except for this (non-physical) case, the above inequality shows that for a wide class of functions, the functional $\Lambda(f)$ is not bounded from below since the control parameter z in (3.5) can have arbitrarily small values.

We postpone the proof of Theorem 1 to Sect. 6. Here we discuss some applications of the inequality (3.5). We first note that $\phi(\mathbf{v}) = \exp(-|\mathbf{v}|^2)$ is a typical function satisfying (3.1)–(3.2). We substitute this function into (3.3) and evaluate the moments $m_s(f)$ when ε given by (3.4). Then we obtain:

$$m_s(f) = \langle f, |\mathbf{v}|^s\rangle = m_s^0(1 + v_0^{3+s}\mathrm{e}^{-zv_0^{2-\gamma}}) \tag{3.6}$$

where m_s^0 denotes the corresponding unperturbed moment. Hence there is no hope to prove the conjectured inequality (1.6) (for $\gamma < 2$) with λ depending on a finite number of moments (of order, say, $s \leqslant n_0$) and bounded from below in a small neighborhood of the unperturbed moments m_s^0. In fact, we can find a z such that the right hand side of the inequality (3.5) is less than any assigned number and then, while keeping z fixed, take v_0 so large as to make the moments arbitrarily close to the unperturbed ones. It is to be remarked that for $v_0^2 > 1$ the function under consideration is not in the L^2 space considered in the previous section.

The inequality (3.5) is also very useful to check any other conjecture concerning lower bounds of $\Lambda(f)$. For example, let us consider the following

Conjecture. The functional $\Lambda(f)$ (for $\gamma < 2$) has a positive lower bound on the set

$$\Omega_\delta^p = \{f(\mathbf{v}) \geqslant 0, \|(f - e^{-|\mathbf{v}|^2})\|_p < \delta\} \tag{3.7}$$

for some $0 < p < 1$ and a sufficiently small δ. Here $\|\cdot\|_p$ is the norm defined by (2.12).

To check this conjecture for a specific value of p, we must verify that all functions defined by (3.3)–(3.4) for sufficiently small values of z and sufficiently large values of v_0 are outside the set Ω_δ^p.

We make again the simplest possible choice, $\phi(\mathbf{v}) = \exp(-|\mathbf{v}|^2)$ in (3.3)–(3.4). We obtain

$$\begin{aligned} \|(f - e^{-|\mathbf{v}|^2})\|_p &= 2\pi \int_0^\infty dx\, x^{1/2} \exp(x^p - x v_0^{-2} - z v_0^{2-\gamma}) \\ &= 2\pi v_0^3 \int_0^\infty dx\, x^{1/2} \exp(x^p v_0^{2p} - x - z v_0^{2-\gamma}), \qquad 0 < p < 1 \end{aligned} \tag{3.8}$$

An elementary estimate, based on the mountain pass method, shows that

$$\|(f - e^{-|\mathbf{v}|^2})\|_p \to 0 \qquad \text{as} \quad v_0 \to \infty \qquad \text{if} \quad p < \frac{2-\gamma}{4-\gamma}$$

We thus obtain the following

Corollary. The conjecture formulated above is certainly not true for $p < (2-\gamma)/(4-\gamma)$, i.e., for $p < 1/2$ in the case of Maxwell molecules ($\gamma = 0$) and $p < 1/3$ in the case of hard spheres ($\gamma = 1$).

Remark 2. The space L_2 with the weighted L_2-norm used in (2.11) roughly corresponds to $p = 1 - \varepsilon$, with ε arbitrarily small, in (3.7). Thus, the functional $\Lambda(f)$ can be bounded from below (with a strictly positive bound) near $e^{-|\mathbf{v}|^2}$ if the perturbations are relatively small in the asymptotic domain $|\mathbf{v}| \to 0$.

In order to prove Theorem 1, we need separate estimates for the functionals $D(f)$ and $\Delta(f)$. We address this matter in the next two sections.

4. AN ESTIMATE OF THE ENTROPY PRODUCTION

It is difficult to obtain a nontrivial lower bound for the non-negative functional $D(f)$. It is much easier to find an upper bound.

We first prove the following simple

Lemma 1. Let $f(\mathbf{v}; \varepsilon, v_0)$ be a function defined in (3.3), where $\phi(\mathbf{v})$ satisfies (3.1). then

$$\log f(\mathbf{v}; \varepsilon, v_0) = \log(a\varepsilon) - b\,\frac{|\mathbf{v}|^2}{v_0^2} + \Psi(\mathbf{v}; \varepsilon, v_0) \tag{4.1}$$

where

$$0 \leqslant \Psi(\mathbf{v}; \varepsilon, v_0) \leqslant \log \frac{1 + \varepsilon\, \|\phi\|_{L^\infty}}{a\varepsilon} + b\,\frac{|\mathbf{v}|^2}{v_0^2} \tag{4.2}$$

the notation being the same as in (3.1).

Proof. In accordance with (3.3) and (4.1):

$$\Psi(\mathbf{v}; \varepsilon, v_0) = \log\left(\frac{\phi(\mathbf{v}/v_0)\, e^{b\,|\mathbf{v}|^2/v_0^2}}{a} + \frac{1}{a\varepsilon}\, e^{((b/v_0^2) - 1)\,|\mathbf{v}|^2}\right) \geqslant 0 \tag{4.3}$$

because of (3.1). On the other hand, using the expression of Ψ which we have just written, we have:

$$\begin{aligned}\Psi(\mathbf{v}; \varepsilon, v_0) &= \frac{b\,|\mathbf{v}|^2}{v_0^2} + \log\left(\frac{\phi(\mathbf{v}/v_0)}{a} + \frac{1}{a\varepsilon}\, e^{-|\mathbf{v}|^2}\right)\\ &\leqslant \frac{b\,|\mathbf{v}|^2}{v_0^2} + \log\left(\frac{\|\phi\|_{L^\infty}}{a} + \frac{1}{a\varepsilon}\right)\end{aligned} \tag{4.4}$$

and (4.2) follows by a simple rearrangement.

The next step requires to split the collision term into the difference of the gain and loss terms $Q_\pm(f, f)$ (both non-negative). The form of these terms is obvious from (1.1) (see also refs. 8 and 9). Moreover, if

$$f(\mathbf{v}) = e^{-|\mathbf{v}|^2} + \tilde{f}(\mathbf{v}), \qquad \tilde{f}(\mathbf{v}) \geqslant 0 \tag{4.5}$$

we obtain a similar representation:

$$Q(f, f) = \tilde{Q}_+(\tilde{f}) - \tilde{Q}_-(\tilde{f}) \tag{4.6}$$

where the modified gain and loss terms

$$\tilde{Q}_\pm(\tilde{f}) = \varepsilon Q_\pm(\tilde{f}, \mathrm{e}^{-|\mathbf{v}|^2}) + \varepsilon Q_\pm(\mathrm{e}^{-|\mathbf{v}|^2}, \tilde{f}) + \varepsilon^2 Q_\pm(\tilde{f}, \tilde{f}) \tag{4.7}$$

are both non-negative because this property has been assumed for $\tilde{f}$. The bilinear operator $Q(f_1, f_2)$ is clearly related to $Q(f, f)$, but, at variance with the most widespread use, is not symmetrized. We remark that we can rewrite the entropy production in terms of the functions f and Ψ defined in (3.3) and (4.1), respectively, as follows:

$$D(f) = -\langle Q(f, f), \log f \rangle = -\langle Q(f, f), \Psi \rangle \tag{4.8}$$

since two more terms appearing in Eq. (4.1) disappear, because of conservation laws. We can rewrite (4.8) as

$$0 \leqslant D(f) = \langle \tilde{Q}_-(\tilde{f}), \Psi \rangle - \langle \tilde{Q}_+(\tilde{f}), \Psi \rangle \tag{4.9}$$

where the notation defined by (4.5)–(4.7) is used with

$$\tilde{f}(\mathbf{v}) = \phi(\mathbf{v}/v_0) \tag{4.10}$$

In accordance with Lemma 1, $\Psi(\mathbf{v}; \varepsilon, v_0)$ is a non-negative function and satisfies the inequality (4.2). This immediately yields an upper estimate for $D(f)$ via Eq. (4.9):

$$0 \leqslant D(f) \leqslant \langle \tilde{Q}_-(\tilde{f}), \Psi \rangle \leqslant \left\langle \tilde{Q}_-(\tilde{f}), L(\varepsilon) + b\, \frac{|\mathbf{v}|^2}{v_0^2} \right\rangle \tag{4.11}$$

where

$$L(\varepsilon) = \log \frac{1 + \varepsilon\, \|\phi\|_{L^\infty}}{a\varepsilon} \tag{4.12}$$

We note that

$$Q_-(f_1, f_2) = f_1 \int_{\mathfrak{R}^3} d\mathbf{w}\, f_2(\mathbf{w})\; g(|\mathbf{v} - \mathbf{w}|) \tag{4.13}$$

and, because of assumption (2.1), $g(|\mathbf{v} - \mathbf{w}|) \leqslant 2g_0$ for $\gamma = 0$ and

$$g(|\mathbf{v} - \mathbf{w}|) \leqslant g_\gamma(1 + |\mathbf{v} - \mathbf{w}|^\gamma) \leqslant 2^{\gamma/2} g_\gamma(1 + |\mathbf{v}|^\gamma + |\mathbf{w}|^\gamma) \tag{4.14}$$

for $0<\gamma\leqslant 2$. Hence the right hand side of the inequality (4.11), where $\tilde{Q}_{-}(\tilde{f})$ is defined by (4.7) and (4.10), can be easily estimated through relatively simple integrals, which may be expressed in terms of moments. These integrals have the general form:

$$I_{i,\,j}(\varepsilon, v_0)=\int_{\mathfrak{R}^3\times\mathfrak{R}^3} d\mathbf{v}\, d\mathbf{w}\, f_i(\mathbf{w})\, f_j(\mathbf{w})(1+|\mathbf{v}|^{\gamma}+|\mathbf{w}|^{\gamma})\left(L(\varepsilon)+b\,\frac{|\mathbf{v}|^2}{v_0^2}\right) \tag{4.15}$$

where $i,\, j=1, 2$ and

$$f_1(\mathbf{v})=e^{-|\mathbf{v}|^2}, \qquad f_2(\mathbf{v})=\varepsilon\phi(\mathbf{v}/v_0) \tag{4.16}$$

The integrals in (4.15) converge because of the assumption (3.2).

We omit the simple calculations and present the final results for the leading terms when $\varepsilon\to 0$ and $v_0\to\infty$:

$$I_{22}=2v_0^{6+\gamma}\left(\varepsilon^2\log\frac{1}{\varepsilon}\right)\langle\phi, 1\rangle\langle\phi, |\mathbf{v}|^{\gamma}\rangle(1+\cdots) \tag{4.17}$$

$$I_{21}=I_{12}=\pi^{3/2}v_0^{3+\gamma}\left(\varepsilon\log\frac{1}{\varepsilon}\right)\langle\phi, |\mathbf{v}|^{\gamma}\rangle(1+\cdots) \tag{4.18}$$

where the dots indicate terms vanishing as $\varepsilon\to 0$ and/or $v_0\to\infty$.

We note that the nonlinear term (4.17) is relatively small if $\varepsilon v_0^3\to 0$, i.e., if the perturbation of the density is negligible. This is the case of interest for our aim; thus our final result involves just the linear terms (4.18).

We can now estimate the right hand side of (4.11) by using the asymptotic behavior of the integrals $I_{i,\,j}$, which we have just found. If we make a minor correction of the numerical coefficient in the case $\gamma=0$, we finally obtain the following

Lemma 2. Let $f(\mathbf{v};\varepsilon, v_0)$ be a function defined by Eqs. (3.1)–(3.3). Then the integral (1.5), giving the entropy production, satisfies the following asymptotic estimate:

$$D(f(\mathbf{v};\varepsilon, v_0))\leqslant 4\pi^{3/2}g_{\gamma}v_0^{3+\gamma}\left(\varepsilon\log\frac{1}{\varepsilon}\right)\langle\phi, |\mathbf{v}|^{\gamma}\rangle(1+\cdots) \tag{4.19}$$

where $0\leqslant\gamma\leqslant 2$ is the same as in (2.1) and dots denote terms vanishing in the limit

$$\varepsilon\to 0, \qquad v_0\to\infty, \qquad \varepsilon v_0^3\to 0 \tag{4.20}$$

5. ASYMPTOTICS OF THE RELATIVE ENTROPY

Our aim in this section is to prove an asymptotic formula for the functional $\Delta(f)$, giving the relative entropy and defined by (1.4). We shall, in fact, prove the following

Lemma 3. If $f(\mathbf{v}; \varepsilon, v_0)$ satisfies the assumptions of Lemma 2, then

$$\Delta(f(\mathbf{v}; \varepsilon, v_0)) = \varepsilon v_0^5 I\left(v_0^{-2} \log \frac{1}{\varepsilon}\right)(1 + \cdots) \tag{5.1}$$

where

$$I(R^2) = \int_{|\mathbf{v}| > R} d\mathbf{v}\, \phi(\mathbf{v})(|\mathbf{v}|^2 - R^2) \tag{5.2}$$

and dots denote terms vanishing in the limit

$$\varepsilon \to 0, \qquad v_0 \to \infty, \qquad \varepsilon v_0^5 \to 0 \tag{5.3}$$

Proof. We fix the function $\phi(\mathbf{v})$ in (3.3) and consider the integral

$$\begin{aligned}\Phi(\varepsilon, v_0) = H(f(\,\cdot\,; \varepsilon, v_0)) - H(e^{-|\mathbf{v}|^2}) &= \langle e^{-|\mathbf{v}|^2}, \log(1 + \varepsilon\phi(\mathbf{v}/v_0)\, e^{|\mathbf{v}|^2})\rangle \\ &\quad + \varepsilon\langle \phi(\mathbf{v}/v_0), \log(e^{-|\mathbf{v}|^2} + \varepsilon\phi(\mathbf{v}/v_0))\rangle \end{aligned} \tag{5.4}$$

The first term in the right hand side is positive and does not exceed $(\varepsilon v_0^3)\langle \phi, 1\rangle$ since $\log(1+x) \leqslant x$. We thus concentrate on the second term and rewrite (5.4) in the form:

$$\Phi(\varepsilon, v_0) = \varepsilon v_0^3 I_1(R^2, v_0^2) + O(\varepsilon v_0^3), \qquad R^2 = v_0^{-2} \log \frac{1}{\varepsilon} \tag{5.5}$$

where

$$\begin{aligned} I_1(R^2, q) &= \langle \phi(\mathbf{v}), \log(e^{-q|\mathbf{v}|^2} + \phi(\mathbf{v})\, e^{-qR^2})\rangle \\ &= \int_{|\mathbf{v}| < R} \phi(\mathbf{v})[\log(1 + \phi(\mathbf{v})\, e^{-q(R^2 - |\mathbf{v}|^2)}) - q\,|\mathbf{v}|^2] \\ &\quad + \int_{|\mathbf{v}| > R} \phi(\mathbf{v})[\log(e^{-q(|\mathbf{v}|^2 - R^2)} + \phi(\mathbf{v})) - qR^2] \end{aligned} \tag{5.6}$$

Hence

$$I_1(R^2, q) = -q\langle \phi(\mathbf{v}), \min(|\mathbf{v}|^2, R^2)\rangle + I_2(R^2, q) \tag{5.7}$$

where

$$|I_2(R^2, q)| \leqslant \langle \phi(\mathbf{v}), \log(1+\phi(\mathbf{v})) + |\log(\mathrm{e}^{-q\,||\mathbf{v}|^2 - R^2|} + \phi(\mathbf{v}))| \rangle \tag{5.8}$$

The function $|I_2(R^2, q)|$ is uniformly bounded because of the elementary inequalities:

$$\begin{aligned} \log \phi \leqslant \log(\phi + s) \leqslant |\log \phi| + \log 2 \\ \log a - b\,|\mathbf{v}|^2 \leqslant \log \phi \leqslant \log \|\phi\|_{L^\infty} \end{aligned} \tag{5.9}$$

which are fulfilled for any number $0 \leqslant s \leqslant 1$ and any function $\phi(\mathbf{v})$ satisfying (3.1)–(3.2). Thus we obtain the asymptotic formula for $\Phi(\varepsilon, v_0)$ defined by (5.5):

$$\Phi(\varepsilon, v_0) = -\varepsilon v_0^5 \langle \phi(\mathbf{v}), \min(|\mathbf{v}|^2, R^2) \rangle + O(\varepsilon v_0^3) \tag{5.10}$$

with $R^2 = v_0^{-2} \log(1/\varepsilon)$. To complete the proof, we need to estimate the function

$$\Phi_M(\varepsilon, v_0) = H(M) - H(\mathrm{e}^{-|\mathbf{v}|^2}) \tag{5.11}$$

where M denotes the Maxwellian which has the same density and temperature as f. Noting that

$$\begin{aligned} m_0 &= \langle f, 1 \rangle = \pi^{3/2} + O(\varepsilon v_0^3) \\ m_2 &= \langle f, |\mathbf{v}|^2 \rangle = \frac{3\pi^{3/2}}{2} + O(\varepsilon v_0^5) \langle \phi, |\mathbf{v}|^2 \rangle \end{aligned}$$

we omit the remaining, simple calculations and give the final asymptotic formula

$$\Phi_M(\varepsilon, v_0) = -\varepsilon v_0^5 \langle \phi(\mathbf{v}), |\mathbf{v}|^2 \rangle + O(\varepsilon v_0^3) + O(\varepsilon^2 v_0^{10}) \tag{5.12}$$

In order to conclude it is enough to note that

$$\Delta(\varepsilon, v_0) = \Phi(\varepsilon, v_0) - \Phi_M(\varepsilon, v_0) \tag{5.13}$$

and use the asymptotic estimates for both $\Phi(\varepsilon, v_0)$ and $\Phi_M(\varepsilon, v_0)$, given by (510) and (5.12).

Remark 3. The asymptotic formula (5.1) can be easily generalized to the case $\varepsilon v_0^5 = \text{const}$. However, our main interest in this paper concerns distribution functions $f(\mathbf{v}; \varepsilon, v_0)$ of the form shown in (3.3) with small perturbations of the second moment (energy). Therefore the result stated in Lemma 3 is sufficient for our goals.

6. THE RATE OF ENTROPY PRODUCTION

We remark that the main results of Sects. 4–5 were proved under the same assumptions on the function $f(\mathbf{v}; \varepsilon, v_0)$. In addition, the assumptions (4.3) in Lemma 3 are stronger than the analogous assumptions (4.20) of Lemma 2. We can thus combine these lemmas and obtain an asymptotic estimate for the functional $\Lambda(f)$, which characterizes the rate of entropy production and was defined in (2.10). A preliminary result concerning $\Lambda(f)$ reads as follows:

Theorem 2. If $f(\mathbf{v}; \varepsilon, v_0)$ satisfies the assumptions of Lemma 2 and

$$\varepsilon \to 0, \qquad v_0 \to \infty, \qquad \varepsilon v_0^5 \to 0 \tag{6.1}$$

then the following asymptotic inequality holds for the non-negative functional $\Lambda(f)$:

$$\Lambda(f(\mathbf{v}; \varepsilon, v_0)) \leqslant 4\pi^{3/2} g_\gamma v_0^{-2+\gamma} \left(\log \frac{1}{\varepsilon}\right) A\left(\phi; \frac{\log(1/\varepsilon)}{v_0^2}\right)(1 + \cdots) \tag{6.2}$$

where

$$A(\phi; R^2) = \frac{\langle \phi, |\mathbf{v}|^\gamma \rangle}{I(R^2)} \tag{6.3}$$

$I(R^2)$ is defined by (5.2) and dots denote terms vanishing in the limit (6.1).

Proof. It is enough to substitute the estimates provided by (4.19) and (5.1) into the definition (2.10).

It is now easy to obtain Theorem 1 formulated in Sect. 3. The proof is a simple consequence of Theorem 2. We consider the inequality (6.2) and express ε in terms of z and v_0 according to the relation (3.4). Then

$$R^2 = v_0^{-2} \log \frac{1}{\varepsilon} = z v_0^{-\gamma} \to 0, \qquad v_0^{\gamma-2} \log \frac{1}{\varepsilon} = z, \qquad \varepsilon v_0^5 = v_0^5\, e^{-z v_0^{2-\gamma}} \to 0$$

for $0 < \gamma < 2$ and any fixed $z > 0$, provided $v_0 \to \infty$.

Then $I(R^2) \to I(0) = \langle \phi, |\mathbf{v}|^2 \rangle$ and we obtain the estimate of Theorem 1, (3.5), from (6.2). In the (non-physical) case $\gamma = 2$ the inequality does not depend on v_0 if $v_0^{-2} \log(1/\varepsilon) \to 0$. Therefore the inequality in Theorem 1, (3.5), does not make sense for $\gamma = 2$: in fact, it holds only for sufficiently large values of z (much larger that $5 \log v_0$). In the case $\gamma = 0$ (Maxwellian molecules), z coincides with R^2 and therefore the estimate in the theorem holds only for sufficiently small values of z; otherwise, we must replace $k(\phi)$ by $A(\phi; z)$ (defined by (6.3)) in (3.5). Thus we have considered all the cases of Theorem 1 and completed its proof.

7. CONCLUDING REMARKS

We have shown that a reasonable conjecture on the rate of entropy production for the space-homogeneous Boltzmann equation is not true even for solutions which are pointwise close to a Maxwellian and have moments of arbitrarily high order close to the corresponding moments of the same Maxwellian. The arguments and estimates used in the paper are of interest in themselves because they provide an asymptotic formula for the rate of entropy production, which may be used to disprove other likely conjectures. An example of application of these estimates has been given.

Future work in this area should be directed toward the assumptions required for the conjecture to be true and a better understanding of the behavior of the function ϕ appearing the Carlen-Carvahlo inequality (1.7) near the origin. Other open questions still remain for the space homogeneous equation. For example, it is clear now that we can have as small entropy production rate at $t = 0$ (by choosing appropriate initial conditions) as we like; moreover the rate will be small during at least some (small) time interval $(0, t_0)$. However what is unclear is the following: is it possible to construct a solution of BE, for which the entropy changes only a little (say, by 0.001 of its initial value) during a given finite time interval $(0, T)$? Our estimates are a first step into the direction of answering this question. Preliminary calculations seem to indicate that the above small change is impossible for a real case of hard spheres or potentials with a compact support, but probably possible for "potentials" (with angular cut-off) softer than hard spheres.

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